Concentration and compactness in nonlinear Schrödinger-Poisson system with a general nonlinearity*

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Abstract

In this paper we use a concentration and compactness argument to prove the existence of a nontrivial nonradial solution to the nonlinear Schrödinger-Poisson equations in $\mathbb{R}^3$, assuming on the nonlinearity the general hypotheses introduced by Berestycki & Lions.

Introduction

We consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + q\phi u = g(x,u) & \text{in } \Omega, \\ -\Delta \phi = qu^2 & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is an unbounded domain in $\mathbb{R}^3$ and $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. In [2] the system has been studied using a variational approach, for $\Omega = \mathbb{R}^3$ and assuming on $g = g(u)$ the Berestycki and Lions hypotheses (see [8]). In particular, it has been showed that the solutions can be found as critical points of an associated functional defined in $H^1(\mathbb{R}^3)$. A first difficulty in applying the classical methods of critical points theory is the lack of compactness, due to the unboundedness of the domain. In [2] this difficulty has been overcome by restricting the functional to the natural constraint

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$H^1_r(\mathbb{R}^3)$, the set of the radially symmetric functions in $H^1(\mathbb{R}^3)$, for which compact embeddings hold.

However, it could happen that such a restriction is not allowed or not suitable to our aim. For example, consider these three situations:

- Ω is not radially symmetric with respect to a point,
- $g(\cdot, s)$ is not invariant under the action of the group of rotations (for example in presence of a breaking-symmetry potential),
- we are looking for non-radial solutions of the problem.

Each of these situations does not allow us to use the set of the radially symmetric functions as a nice functional setting, and we have to handle the problem of the lack of compactness using a different approach.

The aim of this paper is to show how the concentration and compactness principle can be used as an alternative technique to get compactness. In particular, in the same spirit of [11], we are interested in looking for non-radial solutions to the problem

$$\begin{cases} -\Delta u + q\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = qu^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{$\mathcal{SP}$}$$

In [11] an existence result has been proved assuming that $g(u) = |u|^{p-2}u$ and $4 < p < 6$. Here we consider a more general nonlinear term, namely a Berestycki & Lions type nonlinearity. So we assume that

(g1) $g \in C(\mathbb{R}, \mathbb{R})$, g odd;

(g2) $-\infty < \liminf_{s\to 0^+} g(s)/s \leqslant \limsup_{s\to 0^+} g(s)/s = -\omega < 0$;

(g3) $-\infty \leqslant \limsup_{s\to +\infty} g(s)/s^p \leqslant 0$, $1 < p < 5$;

(g4) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\, ds > 0$.

The literature on the Schrödinger-Poisson system in presence of a pure power nonlinearity is very reach: we mention [1, 2] and the references therein. In [9, 10, 23], also the linear and the asymptotic linear case have been studied, whereas in [19, 20, 22] the problem has been studied in a bounded domain. We refer to [6] for more details on the physical origin of this system.

Recently, the Schrödinger equation and the Schrödinger-Poisson system in presence of a general nonlinear term have been intensively studied by many authors. Using similar assumptions on the nonlinearity g,

[4, 14] and [21] studied, respectively, a nonlinear Schrödinger equation in presence of an external potential and a system of weakly coupled nonlinear Schrödinger equations. The Schrödinger-Poisson system has been considered in [2]. We mention also [7, 18] where the Klein-Gordon, Klein-Gordon-Maxwell and Schrödinger Poisson equations have been considered in presence of the so called "positive potentials".

It is well known that the system ($\mathcal{SP}$) is equivalent to an equation containing a nonlocal nonlinear term. A non trivial difficulty in applying concentration and compactness to this equation in presence of a Berestycki & Lions type nonlinearity, consists in the fact that, since g does not have any homogeneity property, we can not use the usual arguments as in the pure power case to avoid dichotomy (see [3]). In order to overcome this difficulty, we need to study the behaviour of the functional associated to the problem with respect to rescaled functions. However, when we rescale the variables, the behaviour of the integral term coming from the nonlocal nonlinearity is such to prevent us from using a direct approach. So we introduce a modified functional, where a cut off function is introduced to control the integral containing the *coupling term*. Finally, we observe that, for q small enough, the modified functional corresponds with the original one computed on suitable minimizing sequences. Observe that, for our analysis, it is fundamental the invariance of the domain with respect to rescalements.

The main result of this paper is the following:

Theorem 0.1. *Assume (**g1**),...,(**g4**). Then there exists $q > 0$ such that the system ($\mathcal{SP}$) possesses a solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ with the following features*

1. *u and ϕ are respectively odd and even with respect to the third variable,*
2. *u and ϕ are cylindrically symmetric with respect to the first two variables,*
3. *u is positive on the half space $x_3 > 0$ (and, consequently, negative in the half space $x_3 < 0$), ϕ is positive everywhere.*

The paper is organized as follows:

in section 1 we introduce the functional framework of the problem. In particular, we define a space of functions described by symmetry properties that no radial nontrivial function possesses. Then we reduce the study to a minimization problem.

In section 2, we study the behaviour of the positive measures associated to the functions of a minimizing sequence, and we look for concentration on a bounded region.

In section 3 we provide the proof of the main theorem.

1 The functional setting

We denote by $H^1(\mathbb{R}^3)$, $\mathcal{D}^{1,2}(\mathbb{R}^3)$, $L^p(\mathbb{R}^3)$ the usual Sobolev and Lebesgue spaces with the respective norms:

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\right)^{\frac{1}{2}}$$

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}}$$

$$\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p\right)^{\frac{1}{p}}.$$

We first recall the following well-known facts (see, for instance [12]).

Lemma 1.1. *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = qu^2, \qquad \text{in } \mathbb{R}^3.$$

Moreover

i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = q\int_{\mathbb{R}^3} \phi_u u^2$;

ii) $\phi_u \geqslant 0$;

iii) for any $\theta > 0$: $\phi_{u_\theta}(x) = \theta^2\phi_u(x/\theta)$, where $u_\theta(x) = u(x/\theta)$;

iv) there exist $C, C' > 0$ independent of $u \in H^1(\mathbb{R}^3)$ such that

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant Cq\|u\|^2,$$

and

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C'q\|u\|^4. \tag{2}$$

Following [8], define $s_0 := \min\{s \in [\zeta, +\infty[\mid g(s) = 0\}$ ($s_0 = +\infty$ if $g(s) \neq 0$ for any $s \geqslant \zeta$) and set $\tilde{g} : \mathbb{R} \to \mathbb{R}$ the function such that

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0]; \\ 0 & \text{on } \mathbb{R}_+ \setminus [0, s_0]; \\ -\tilde{g}(-s) & \text{on } \mathbb{R}_-. \end{cases} \tag{3}$$

By the strong maximum principle and by ii) of Lemma 1.1, a solution of $(\mathcal{SP})$ with $\tilde{g}$ in the place of g is a solution of $(\mathcal{SP})$. So we can suppose that

g is defined as in (3), so that (**g1**), (**g2**) and (**g4**) hold, and we have also the following limit

$$\lim_{s\to\infty}\frac{|g(s)|}{|s|^p}=0. \tag{4}$$

Moreover, we set for any $s\geqslant 0$,

$$g_1(s):=(g(s)+\omega s)^+,$$
$$g_2(s):=g_1(s)-g(s),$$

and we extend them as odd functions.
Since

$$\lim_{s\to 0}\frac{g_1(s)}{s}=0,$$
$$\lim_{s\to\infty}\frac{g_1(s)}{|s|^p}=0, \tag{5}$$

and

$$g_2(s)\geqslant \omega s,\quad \forall s\geqslant 0, \tag{6}$$

by some computations, we have that for any $\varepsilon>0$ there exist $C_\varepsilon, C'_\varepsilon>0$ such that

$$g_1(s)\leqslant C_\varepsilon s^p+\varepsilon s,\quad \forall s\geqslant 0 \tag{7}$$
$$g_1(s)\leqslant C'_\varepsilon s^5+\varepsilon s,\quad \forall s\geqslant 0 \tag{8}$$
$$g_1(s)\leqslant C_\varepsilon s^p+\varepsilon g_2(s),\quad \forall s\geqslant 0 \tag{9}$$
$$g_1(s)\leqslant C'_\varepsilon s^5+\varepsilon g_2(s),\quad \forall s\geqslant 0. \tag{10}$$

If we set

$$G_i(t):=\int_0^t g_i(s)\,ds,\quad i=1,2,$$

then, by (6), we have

$$G_2(s)\geqslant \frac{\omega}{2}s^2,\quad \forall s\in\mathbb{R} \tag{11}$$

and by (7), (8), (9) and (10), for any $\varepsilon>0$ there exists $C_\varepsilon>0$ and $C'_\varepsilon>0$ such that

$$G_1(s)\leqslant \frac{C_\varepsilon}{6}|s|^6+\varepsilon s^2,\quad \forall s\in\mathbb{R}$$
$$G_1(s)\leqslant \frac{C'_\varepsilon}{p+1}|s|^{p+1}+\varepsilon s^2,\quad \forall s\in\mathbb{R} \tag{12}$$
$$G_1(s)\leqslant \frac{C_\varepsilon}{6}|s|^6+\varepsilon G_2(s),\quad \forall s\in\mathbb{R} \tag{13}$$
$$G_1(s)\leqslant \frac{C'_\varepsilon}{p+1}|s|^{p+1}+\varepsilon G_2(s),\quad \forall s\in\mathbb{R}. \tag{14}$$

The solutions $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ of ($\mathcal{SP}$) are the critical points of the action functional $\mathcal{E}_q\colon H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)\to\mathbb{R}$, defined as

$$\mathcal{E}_q(u,\phi):=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2-\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2+\frac{q}{2}\int_{\mathbb{R}^3}\phi u^2-\int_{\mathbb{R}^3}G(u).$$

The action functional $\mathcal{E}_q$ is strongly indefinite in the sense that it is unbounded both from below and from above on infinite dimensional subspaces. The indefiniteness can be removed using the reduction method, by which we are led to study a one variable functional that does not present such a strongly indefinite nature. Indeed, it can be proved that $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of ($\mathcal{SP}$) (critical point of functional $\mathcal{E}_q$) if and only if $u\in H^1(\mathbb{R}^3)$ is a critical point of the functional $J_q\colon H^1(\mathbb{R}^3)\to\mathbb{R}$ defined as

$$J_q(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{q}{4}\int_{\mathbb{R}^3}\phi_u u^2-\int_{\mathbb{R}^3}G(u),$$

and $\phi=\phi_u$.

Now, let $\mathcal{O}(2)$ denote the orthogonal group of the rotation matrices in $\mathbb{R}^2$, that is

$$\mathcal{O}(2)=\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\Big|\,\alpha\in[0,2\pi)\right\}.$$

For any $g\in\mathcal{O}(2)$ define the following action $\mathcal{T}_g$ on $H^1(\mathbb{R}^3)$:

$$\mathcal{T}_g u(x)=-u(\tilde{g}x)\in H^1(\mathbb{R}^3),\quad \tilde{g}=\begin{pmatrix}g & 0\\ 0 & -1\end{pmatrix}.$$

Now we set

$$H^1_{cyl,o}(\mathbb{R}^3)=\{u\in\mathcal{D}^1(\mathbb{R}^3,\mathbb{R}^3)\mid\mathcal{T}_g u=u\ \ \forall g\in\mathcal{O}(2)\}.$$

It is easy to see that $H^1_{cyl,o}(\mathbb{R}^3)$ is the setting of the functions cylindrically symmetric with respect to (x_1,x_2) and odd with respect to x_3.

Since g is odd (and consequently G is even) and since we have that for any $u\in H^1(\mathbb{R}^3)$ and $g\in\mathcal{O}(2)$

$$-\mathcal{T}_g\phi_u=\phi_{\mathcal{T}_g u}\tag{15}$$

by the Palais' symmetrical criticality principle we can prove that $H^1_{cyl,o}(\mathbb{R}^3)$ is a natural constraint for the action functional J_q (see [11] for details).
We point out that, since $u\in H^1_{cyl,o}(\mathbb{R}^3)$, we have that $\phi_u\in\mathcal{D}^{1,2}_{cyl,e}(\mathbb{R}^3)$, the set of the functions in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ that are cylindrically symmetric with respect to the first two variables, and even with respect to the third. To improve the notations, we will often use r in the place of $\sqrt{x_1^2+x_2^2}$.

We will proceed as follows: we consider the manifold

$$\mathcal{M} = \{u \in H^1_{cyl,o}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} G(u) = 1\}. \tag{16}$$

As proved in [5] (see also [8]), $\mathcal{M}$ is nonempty. Consider indeed a family of functions $\rho_R(r, x_3) = \xi\alpha_R(r)\beta_R(x_3)$, for $R > 1$, with

$$\alpha_R(t) := \begin{cases} 1 & \text{if} \quad |t| < R, \\ R+1-|t| & \text{if} \quad R \leqslant |t| < R+1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\beta_R(t) := \begin{cases} 0 & \text{if} \quad 0 < t \leqslant 1 \\ t-1 & \text{if} \quad 1 < t \leqslant 2, \\ 1 & \text{if} \quad 2 < t \leqslant R, \\ R+1-t & \text{if} \quad R < t \leqslant R+1, \\ -\beta_R(-t) & \text{if} \quad t \leqslant 0. \end{cases}$$

We have $\rho_R \in H^1_{cyl,o}(\mathbb{R}^3)$, and for large $\bar{R}$

$$\int_{\mathbb{R}^3} G(\rho_{\bar{R}}) > 0.$$

So, if σ is a suitable rescaling parameter, the function

$$\rho_{\bar{R},\sigma} : (r, x_3) \mapsto \rho_{\bar{R}}(\sigma r, \sigma x_3)$$

belongs to $\mathcal{M}$.

Then, we consider the functional

$$J_q(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}\int_{\mathbb{R}^3} \phi_u u^2 \tag{17}$$

restricted on $\mathcal{M}$, and we look for a minimizer $\bar{u}$.
Solving the minimizing problem, we find a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the tern $(\bar{u}, \phi_{\bar{u}}, \lambda)$ solves the system

$$\begin{cases} -\Delta u + q\phi u = \mu g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = qu^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Then we apply the following

Theorem 1.2. *Let $\bar{u} \in \mathcal{M}$ a minimizer for $J_q|_{\mathcal{M}}$, and let λ be the Lagrange multiplier. Then λ is positive, and the couple $(\tilde{u}, \tilde{\phi}) \in H^1_{cyl,o}(\mathbb{R}^3) \times \mathcal{D}^{1,2}_{cyl,e}(\mathbb{R}^3)$ defined rescaling as follows*

$$\tilde{u} = \bar{u}(\cdot/\sqrt{\lambda}) \qquad \tilde{\phi} = \phi_{\bar{u}}(\cdot/\sqrt{\lambda}) \tag{18}$$

solves the system

$$\begin{cases} -\Delta u + q'\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = q' u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{19}$$

with $q' = q/\lambda$.

2 Compactness

In this section we present the main tool to get our result. We first need to introduce some notations and definitions.
Set $m_q = \inf_{u\in\mathcal{M}} J_q(u)$, and denote by $(u_n)_n := (u_n^q)_n$ a sequence such that

$$u_n \in \mathcal{M} \qquad \text{and} \qquad J_q(u_n) \to m_q \tag{20}$$

and by $\phi_n = \phi_{u_n}$.

As in [2, 13, 15] we introduce the cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \chi(s) = 1, & \text{for } s \in [0,1], \\ 0 \leqslant \chi(s) \leqslant 1, & \text{for } s \in]1,2[, \\ \chi(s) = 0, & \text{for } s \in [2,+\infty[, \\ \|\chi'\|_\infty \leqslant 2, & \end{cases} \tag{21}$$

and, for every $T > 0$, we denote

$$k_T(u) = \chi\left(\frac{\|u\|^2}{T^2}\right).$$

Moreover, assume the following definitions

$$J_q^T(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} k_T(u) \int_{\mathbb{R}^3} \phi_u u^2$$
$$\mu_n^{T,q}(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \int_\Omega G_2(u_n) + \frac{q}{4} k_T(u_n) \int_\Omega \phi_n u_n^2,$$

where $\Omega \subset \mathbb{R}^3$. Set also $m_q^T = \inf_{u\in\mathcal{M}} J_q^T(u)$, and denote by $(u_n^{T,q})_n$ a minimizing sequence of $J_q^T|_{\mathcal{M}}$. It is trivial to see that $m_q^T \leqslant m_q \leqslant m_{\bar q}$ for any $T > 0$ and any $q \leqslant \bar q$.

Lemma 2.1. *For any $T, q > 0$ the measures $\mu_n^{T,q}$ are positive and bounded, i.e. $(\mu_n^{T,q}(\mathbb{R}^3))_n$ is bounded. Moreover $\mu_n^{T,q}$ is bounded $T-$uniformly.*

Proof The positiveness is a trivial consequence of the definition of the measures.

As to boundedness, by the very definition of u_n we have only to check if $(\int_{\mathbb{R}^3} G_2(u_n))_n$ is bounded. But by (13) we have

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \leq \int_{\mathbb{R}^3} \varepsilon G_2(u_n) + C \int_{\mathbb{R}^3} |u_n|^6 \tag{22}$$

and then

$$1 + (1-\varepsilon) \int_{\mathbb{R}^3} G_2(u_n) \leq C' \left(\int_{\Omega} |\nabla u_n|^2 \right)^3 \tag{23}$$

for $0 < \varepsilon < 1$ and C, C' suitable positive constants.
The $T-$uniform boundedness is a consequence of the fact that for any $n \geqslant 1$ and for any $T > 0$ $k_T(u_n) \leq 1$. □

Let $c = c_q^T$ be the limit (up to a subsequence) of $\mu_n^{T,q}(\mathbb{R}^3)$. Of course $c > 0$ because, otherwise, we would contradict (23).

Lemma 2.2. *For any $\bar{q}$ there exists $\bar{T}$ such that*

$$\limsup_n \|u_n^q\| \leq T, \quad \limsup_n \|u_n^{T,q}\| \leqslant T \tag{24}$$

for all $q \leqslant \bar{q}$ and $T \geqslant \bar{T}$.
As a consequence, every a minimizing sequence for $J_q|_{\mathcal{M}}$, is a minimizing sequence also for $J_q^T|_{\mathcal{M}}$.

Proof Fix $\bar{q} > 0$ and $q \leq \bar{q}$ and consider a minimizing sequence $u_n = u_n^q$ as in (20). Consider also $\bar{T} > 0$ whose precise estimate will be given later, $T \geqslant \bar{T}$ and $(u_n^{T,q})_n$ a minimizing sequence of $J_q^T|_{\mathcal{M}}$. Certainly we have that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \leqslant 2m_q + o_n(1) \leqslant 2m_{\bar{q}} + o_n(1). \tag{25}$$

By (11) and (23) we have also

$$\begin{aligned} \int_{\mathbb{R}^3} |u_n|^2 &\leqslant \frac{\omega}{2} \int_{\mathbb{R}^3} G_2(u_n) \leqslant C\big(\int_{\mathbb{R}^3} |\nabla u_n|^2\big)^3 \\ &\leqslant C'(2m_q + o_n(1))^3 = 8C' m_q^3 + o_n(1) \leqslant 8C' m_{\bar{q}}^3 + o_n(1). \end{aligned} \tag{26}$$

Since $m_q^T \leqslant m_q$, the same estimates can be proved also for $(u_n^{T,q})_n$. By (25) and (26) we conclude the first part of the proof taking $\bar{T} > \max(2m_{\bar{q}}, 8C' m_{\bar{q}}^3)$.

To prove the final part of the theorem, it is sufficient to show that $m_q^T = m_q$. But for a sufficiently large $\nu \geqslant 1$ and any $n \geqslant \nu$, by (24) we have that $k_T(u_n^{T,q}) = 1$ and $J_q^T(u_n^{T,q}) = J_q(u_n^{T,q}) \geqslant m_q$. We deduce that $m_q^T \geqslant m_q$ and then $m_q^T = m_q$.

□

By the concentration and compactness principle (see [16]), one of the following holds:

vanishing: for all $R > 0$

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_R(\xi)} d\mu_n^{T,q} = 0;$$

dichotomy: for a subsequence of $(\mu_n^{T,q})_n$, there exist a constant $\tilde{c} \in (0, c)$, $R > 0$, two sequences $(\xi_n)_n$ and $(R_n)_n$, with $R \leqslant R_n$ for any n and $R_n \to +\infty$, such that

$$\int_{B_R(\xi_n)} d\mu_n^{T,q} \to \tilde{c}, \quad \int_{\mathbb{R}^3 \setminus B_{R_n}(\xi_n)} d\mu_n^{T,q} \to c - \tilde{c}, \tag{27}$$

compactness: there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^3$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_R(\xi_n)} d\mu_n \geqslant c - \delta.$$

Theorem 2.3. *Vanishing does not occur*

Proof

Suppose by contradiction, that for all $R > 0$

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_R(\xi)} d\mu_n^{T,q} = 0.$$

In particular, we deduce that there exists $\bar{R} > 0$ such that

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_{\bar{R}}(\xi)} u_n^2 = 0.$$

By this and Lemma 2.2, we have that $u_n \to 0$ in $L^s(\mathbb{R}^3)$, for $2 < s < 6$ (see [17, Lemma I.1]). As a consequence, since $(u_n)_n \subset \mathcal{M}$ and by (14), we get for $0 < \varepsilon < 1$ and $C'_\varepsilon > 0$

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \leq \int_{\mathbb{R}^3} \varepsilon G_2(u_n) + C'_\varepsilon \int_{\mathbb{R}^3} |u_n|^{p+1}$$

and then
$$1 + (1-\varepsilon)\int_{\mathbb{R}^3} G_2(u_n) \le C'_\varepsilon \int_{\mathbb{R}^3} |u_n|^{p+1} \to 0.$$

□

From now on, if the notation of a ball does not present explicitly expressed the center, than we assume it is the origin.

Theorem 2.4. *For any $\bar{q} > 0$, there exist $\bar{T} > 0$ such that for any $T \geqslant \bar{T}$ and a suitable $0 < q(T) \leqslant \bar{q}$, either $\mu_n^{T,q(T)}$ concentrates in a ball B_R (namely compactness holds for $\xi_n = (0,0,0), n \geqslant 1$) or it exhibits the following dichotomic behaviour: there exist $R > 0$ and a divergent sequence $\xi_n = (0,0,x_3^n)_n$ in $\mathbb{R}^3$ such that*

$$\int_{B_R(\xi_n)} d\mu_n^{T,q} \to \frac{c}{2}$$
$$\int_{B_R(-\xi_n)} d\mu_n^{T,q} \to \frac{c}{2}.$$

Proof Take $\bar{q} > 0$, and let $\bar{T} > 0$ be as in Lemma 2.2.
Set $T \geqslant \bar{T}$. Suppose that dichotomy holds and let $\tilde{c} \in (0,c)$, $R > 0$, $(\xi_n)_n$, $(R_n)_n$ be as in the dichotomy hypothesis. We prove that $(\xi_n)_n$ is bounded with respect to the first two variables. Otherwise, we should have $\xi_n \simeq (r_n, x_3^n)$ with $r_n \to +\infty$ and

$$\int_{B_R(\xi_n)} d\mu_n^{T,q} = \tilde{c} + o_n(1). \tag{28}$$

We deduce that there exists a positive constant $C >$ such that

$$\int_{B_R(\xi_n)} |\nabla u_n|^2 + \int_{B_R(\xi_n)} G_2(u_n) \geqslant C$$

(otherwise, by (2) and (11), we would get a contradiction with (28)). But, for r_n that goes to infinity, the set $B_{\xi_n+R}(0) \setminus B_{\xi_n-R}(0)$ contains an increasing number of disjoint balls of the type $B_R(r', x_3^n)$, with $r_n = r' := \sqrt{(x_1')^2 + (x_2')^2}$ and, by the symmetry properties on u_n, for any $n \geqslant 1$,

$$\int_{B_R((r',x_n^3))} |\nabla u_n|^2 + \int_{B_R((r',x_n^3))} G_2(u_n) = \int_{B_R(\xi_n)} |\nabla u_n|^2 + \int_{B_R(\xi_n)} G_2(u_n).$$

As a consequence, we would have that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} G_2(u_n) \to +\infty$$

that, taking (23) into account, brings a contradiction to Lemma 2.2.

By the boundedness of $(\xi_n)_n$ with respect to r_n, it is not restrictive to suppose that such a sequence belongs to the x_3-axis. Indeed, for any $n \geqslant 1$, the ball $B_R(\xi_n)$ is contained in $B_{R'}((0,0,x_3^n))$, where $R' = R + \sup_n |r_n|$.

Now we consider the following possibilities:

- $(x_n^3)_n$ is bounded
- $(x_n^3)_n$ is unbounded.

If $(x_n^3)_n$ is bounded, all the balls of the type $B_R(\xi_n)$ are contained in $B_{R''}$, where $R'' = R' + \sup_n |x_n^3|$. Replacing R' by R'', we have that

$$\int_{B_{R''}(0)} d\mu_n^{T,q} = \tilde{c} + o_n(1). \tag{29}$$

Consider a sequence of radially symmetric cut-off functions $\rho_n \in C^1(\mathbb{R}^3)$ such that $\rho_n \equiv 1$ in $B_R(0)$, $\rho_n \equiv 0$ in $\mathbb{R}^3 \setminus B_{R_n}(0)$, $0 \leqslant \rho_n \leqslant 1$ and $|\nabla \rho_n| \leqslant 2/(R_n - R)$.
We set

$$v_n := \rho_n u_n, \qquad w_n := (1-\rho_n) u_n.$$

Certainly v_n and w_n are in $H^1_{cyl,o}(\mathbb{R}^3)$ and

$$\|v_n\| \leqslant \|u_n\| + o_n(1) \tag{30}$$
$$\|w_n\| \leqslant \|u_n\| + o_n(1). \tag{31}$$

If we denote $\Omega_n := B_{R_n} \setminus B_R$, by dichotomy hypothesis we deduce that

$$\int_{\Omega_n} |\nabla u_n|^2 \to 0, \quad \int_{\Omega_n} G_2(u_n) \to 0, \quad \int_{\Omega_n} \phi_n u_n^2 \to 0, \tag{32}$$

and, in particular,

$$\|u_n\|_{H^1(\Omega_n)} \to 0. \tag{33}$$

Since for suitable ε, C_ε, and $C' > 0$

$$\begin{aligned}\int_{\Omega_n} G_1(u_n) &\leqslant \varepsilon \int_{\Omega_n} G_2(u_n) + C_\varepsilon \int_{\Omega_n} |u_n|^{p+1} \\ &\leqslant \varepsilon \int_{\Omega_n} G_2(u_n) + C' \|u_n\|^{p+1}_{H^1(\Omega_n)},\end{aligned} \tag{34}$$

we have also that

$$\int_{\Omega_n} G_1(u_n) \to 0. \tag{35}$$

Since by simple computations, using (32), we have

$$\int_{\Omega_n} |\nabla v_n|^2 \to 0 \text{ and } \int_{\Omega_n} |\nabla w_n|^2 \to 0,$$

we easily infer that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 = \int_{\mathbb{R}^3} |\nabla v_n|^2 + \int_{\mathbb{R}^3} |\nabla w_n|^2 + o_n(1). \tag{36}$$

Moreover we can prove also that

$$\int_{\Omega_n} G_1(v_n) \to 0 \quad \int_{\Omega_n} G_2(v_n) \to 0 \tag{37}$$

$$\int_{\Omega_n} G_1(w_n) \to 0 \quad \int_{\Omega_n} G_2(w_n) \to 0 \tag{38}$$

Indeed, by (12), the growth conditions on g and (33),

$$\begin{aligned}
\int_{\Omega_n} G_1(v_n) &\leqslant C(\int_{\Omega_n} |v_n|^2 + \int_{\Omega_n} |v_n|^{p+1}) \\
&\leqslant C'(\|v_n\|^2_{H^1(\Omega_n)} + \|v_n\|^{p+1}_{H^1(\Omega_n)}) \\
&\leqslant C'(\|u_n\|^2_{H^1(\Omega_n)} + \|u_n\|^{p+1}_{H^1(\Omega_n)} + o_n(1)) = o_n(1) \\
\int_{\Omega_n} G_2(v_n) &= -\int_{\Omega_n} G(v_n) + \int_{\Omega_n} G_1(v_n) \\
&\leqslant C(\int_{\Omega_n} |v_n|^2 + \int_{\Omega_n} |v_n|^{p+1}) \leqslant C'(\|v_n\|^2_{H^1(\Omega_n)} + \|v_n\|^{p+1}_{H^1(\Omega_n)}) \\
&\leqslant C'(\|u_n\|^2_{H^1(\Omega_n)} + \|u_n\|^{p+1}_{H^1(\Omega_n)} + o_n(1)) \leqslant o_n(1),
\end{aligned}$$

and we proceed analogously for w_n. By (32), (35), (37) and (38), we deduce that

$$\int_{\mathbb{R}^3} G_i(u_n) = \int_{\mathbb{R}^3} G_i(v_n) + \int_{\mathbb{R}^3} G_i(w_n) + o_n(1), \quad i = 1, 2. \tag{39}$$

Finally, as in [3], we have

$$\int_{\mathbb{R}^3} \phi_n u_n^2 \geqslant \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + o_n(1). \tag{40}$$

By (36), (39) and (40), taking into account that by (30), (31) and Lemma 2.2 we have $1 = k_T(u_n) = k_T(v_n) = k_T(w_n)$, we deduce that

$$m_q^T = J_q^T(u_n) + o_n(1) \geqslant J_q^T(v_n) + J_q^T(w_n) + o_n(1)$$

and, as a consequence,

$$\begin{aligned} J_q^T(v_n) &\to \widetilde{m_q^T} \\ J_q^T(w_n) &\to \overline{m_q^T} \end{aligned} \tag{41}$$

with $\widetilde{m_q^T} + \overline{m_q^T} \leqslant m_q^T$.

For the moment, we assume that $\widetilde{m_q^T} \neq 0$ and $\overline{m_q^T} \neq 0$.

We have to consider the following possibilities

i) there exists $0 < \lambda < 1$ *such that, up to subsequences,*

$$\begin{aligned} \int_{\mathbb{R}^3} G(v_n) &\to \lambda \\ \int_{\mathbb{R}^3} G(w_n) &\to 1 - \lambda. \end{aligned}$$

Consider the rescaled functions so defined: $\tilde{v}_n(\cdot) = v_n(\sqrt[3]{\lambda}\,\cdot)$ and $\tilde{w}_n(\cdot) = w_n(\sqrt[3]{1-\lambda}\,\cdot)$ so that we respectively have

$$\begin{aligned} J_q^T(\tilde{v}_n) &\geqslant m_q^T + o_n(1) \\ J_q^T(\tilde{w}_n) &\geqslant m_q^T + o_n(1). \end{aligned}$$

The following chain of inequalities holds

$$\begin{aligned} o_n(1) + \widetilde{m_q^T} &= J_q^T(v_n) \\ &= \frac{\sqrt[3]{\lambda}}{2} \int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 \\ &\quad + \frac{q(\sqrt[3]{\lambda})^5}{4} \chi \left(\frac{(\sqrt[3]{\lambda})^2 \|\nabla \tilde{v}_n\|_2^2 + \lambda^2 \|\tilde{v}_n\|_2^2}{T^2} \right) \int_{\mathbb{R}^3} \phi_{\tilde{v}_n} \tilde{v}_n^2 \\ &\geqslant \frac{\sqrt[3]{\lambda}}{2} \int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 + \frac{q(\sqrt[3]{\lambda})^5}{4} \chi \left(\frac{\|\tilde{v}_n\|^2}{T^2} \right) \int_{\mathbb{R}^3} \phi_{\tilde{v}_n} \tilde{v}_n^2 \\ &\geqslant \lambda m_q^T + \left(\frac{\sqrt[3]{\lambda} - \lambda}{2} \right) \int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 \\ &\quad + \frac{q(\sqrt[3]{\lambda^5} - \lambda)}{4} \chi \left(\frac{\|\tilde{v}_n\|^2}{T^2} \right) \int_{\mathbb{R}^3} \phi_{\tilde{v}_n} \tilde{v}_n^2 + o_n(1). \end{aligned} \tag{42}$$

Now observe that, since $\int_{\mathbb{R}^3} G(\tilde{v}_n) \to 1$, computing as in (22) and (23),

$$o_n(1) + 1 + (1-\varepsilon) \int_{\mathbb{R}^3} G_2(\tilde{v}_n) \leqslant C \left(\int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 \right)^3, \tag{43}$$

for $0<\varepsilon<1$, we deduce that $\|\nabla \tilde v_n\|_2$ is bounded below by a positive constant. Moreover, by (2),

$$\chi\left(\frac{\|\tilde v_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{\tilde v_n}\tilde v_n^2 \leqslant qCT^4, \tag{44}$$

so by (42), (43) and (44), for suitable $a,b>0$, we have

$$\widetilde{m_q^T} \geqslant \lambda m_q^T + a(\sqrt[3]{\lambda}-\lambda)+bq^2(\sqrt[3]{\lambda^5}-\lambda)T^4. \tag{45}$$

But

$$\begin{aligned} a(\sqrt[3]{\lambda}-\lambda)+bq^2(\sqrt[3]{\lambda^5}-\lambda)T^4 &= \sqrt[3]{\lambda}(1-\sqrt[3]{\lambda^2})(a-bq^2\lambda T^4)\\ &\geqslant \sqrt[3]{\lambda}(1-\sqrt[3]{\lambda^2})(a-bq^2T^4), \end{aligned}$$

so, if we take $q<\sqrt{\frac{a}{bT^4}}$, from (45) we obtain $\widetilde{m_q^T}>\lambda m_q^T$.
Repeating the same computations with $\tilde w_n$ in the place of $\tilde v_n$, we can prove that $\overline{m_q^T}>(1-\lambda)m_q^T$. Summing up, we get

$$m_q^T \geqslant \widetilde{m_q^T}+\overline{m_q^T} > \lambda m_q^T + (1-\lambda)m_q^T = m_q^T$$

and then a contradiction.

ii) there exists $\lambda\geqslant 1$ such that, up to subsequences,

$$\int_{\mathbb{R}^3} G(v_n)\to\lambda$$

or

$$\int_{\mathbb{R}^3} G(w_n)\to\lambda.$$

Suppose that the first holds, and set $\lambda_n=\int_{\mathbb{R}^3}G(v_n)$ and $\tilde v_n = v_n(\sqrt[3]{\lambda_n}\,\cdot)\in\mathcal{M}$. We would have the following chain of inequalities

$$\begin{aligned} m_q^T \leqslant J_q^T(\tilde v_n) &= \frac{1}{2\sqrt[3]{\lambda_n}}\int_{\mathbb{R}^3}|\nabla v_n|^2+\frac{q}{4\sqrt[3]{\lambda_n^5}}k_T(\tilde v_n)\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2+\frac{q}{4}k_T(v_n)\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 \to \widetilde{m_q^T} < m_q^T, \end{aligned} \tag{46}$$

where we have used the fact that $\|\tilde v_n\|^2\leqslant\|v_n\|^2\leqslant\|u_n\|^2+o_n(1)<T^2$ to deduce that $k_T(\tilde v_n)=k_T(v_n)=1$.

Now we remove the assumption that $\widetilde{m_q^T} \neq 0$ and $\overline{m_q^T} \neq 0$. If, for instance, $\overline{m_q^T} = 0$, from (27) and (41) we would deduce that

$$\int_{\mathbb{R}^3} |\nabla w_n|^2 \to 0$$
$$\int_{\mathbb{R}^3} G_2(w_n) \geqslant \alpha + o_n(1)$$

with $\alpha > 0$.
Hence, by (14), for any $\varepsilon > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^3} G_1(w_n) &< \varepsilon \int_{\mathbb{R}^3} G_2(w_n) + C_\varepsilon \int_{\mathbb{R}^3} |\nabla w_n|^2 \\ &= \varepsilon\alpha + C_\varepsilon o_n(1) + o_n(1),\end{aligned}$$

and then $\int_{\mathbb{R}^3} G_1(w_n) \to 0$. So

$$\begin{aligned}1 = \int_{\mathbb{R}^3} G(u_n) &= \int_{\mathbb{R}^3} G(v_n) + \int_{\mathbb{R}^3} G(w_n) + o_n(1) \\ &= \int_{\mathbb{R}^3} G(v_n) - \int_{\mathbb{R}^3} G_2(w_n) + o_n(1) \\ &\leqslant \int_{\mathbb{R}^3} G(v_n) - \alpha + o_n(1)\end{aligned}$$

which implies that, up to subsequences, $\int_{\mathbb{R}^3} G(v_n) \to \lambda > 1$.
As in (46),

$$\begin{aligned}m_q^T &\leqslant \liminf_n \left(\frac{1}{2\sqrt[3]{\lambda_n}} \int_{\mathbb{R}^3} |\nabla v_n|^2 + \frac{q}{4\sqrt[3]{\lambda_n^5}} k_T(\tilde{v}_n) \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \right) \\ &\leqslant \liminf_n \left(\frac{1}{2\sqrt[3]{\lambda}} \int_{\mathbb{R}^3} |\nabla v_n|^2 + \frac{q}{4\sqrt[3]{\lambda^5}} k_T(v_n) \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \right) \\ &< \liminf_n \left(\frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_n|^2 + \frac{q}{4} k_T(v_n) \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \right) = \widetilde{m_q^T} = m_q^T\end{aligned}$$

and then a contradiction. The case $\widetilde{m_q^T} = 0$ is analogous.

We have showed that, in any case, if $(x_3^n)_n$ is bounded, dichotomy leads to a contradiction.
It remain to study what would happen if $(x_3^n)_n$ was unbounded. Suppose that the dichotomic behaviour of the statement does not hold. Then, by

the evenness of the functional and the oddness with respect to the third variable of the functions in $H^1_{cyl,o}(\mathbb{R}^3)$,

$$\int_{B_R(\xi_n)} d\mu_n^{T,q} \to \tilde{c} < \frac{c}{2}$$
$$\int_{B_R(-\xi_n)} d\mu_n^{T,q} \to \tilde{c} < \frac{c}{2}$$
$$\int_{\mathbb{R}^3\setminus\Sigma_n} d\mu_n^{T,q} \to c - 2\tilde{c}$$

where we have assumed the following notation: $\Sigma_n = B_{R_n}(\xi_n) \cup B_{R_n}(-\xi_n)$. Observe that we can redefine the sequence R_n in such a way we have $B_{R_n}(\xi_n) \cap B_{R_n}(-\xi_n) = \emptyset$.
Now, consider a sequence of ξ_n-radially symmetric cut-off functions $\rho_n \in C^1(\{x \in \mathbb{R}^3 \mid x_3 > 0\})$ such that $\rho_n \equiv 1$ in $B_R(\xi_n)$, $\rho_n \equiv 0$ in $\{x \in \mathbb{R}^3 \mid x_3 > 0\} \setminus B_{R_n}(0)$, $0 \leqslant \rho_n \leqslant 1$ and $|\nabla \rho_n| \leqslant 2/(R_n - R)$, and define $\sigma_n \in C^1(\mathbb{R}^3)$ by evenness with respect to the third variable.
Set $v_n = \sigma_n u_n$ and $w_n = (1 - \sigma_n)u_n$. Of course v_n and w_n are in $H^1_{cyl,o}(\mathbb{R}^3)$ and we can repeat exactly the same arguments as in the x_3^n *bounded case* to get a contradiction.

The proposition is so completely proved. □

3 Proof of the main Theorem

From now on, all the sequences considered have their $\limsup$ in the norm of $H^1(\mathbb{R}^3)$ less than $\bar{T}$, being $\bar{T}$ the same as in Lemma 2.2. Therefore there is no difference between J_q and J_q^T evaluated on them.

Theorem 3.1. *Let q be as in Theorem 2.4, then the infimum m_q is achieved.*

Proof Suppose that the dichotomy situation described in Theorem 2.4 holds. Since $x_3^n \to +\infty$, we can suppose that for any $n \geqslant 1$ we have $x_3^n > 3R$. Then, consider a sequence of ξ_n-radially symmetric cut-off functions $\rho_n \in C^1(\{x \in \mathbb{R}^3 \mid x_3 > 0\})$ such that $\rho_n \equiv 1$ in $B_R(\xi_n)$, $\rho_n \equiv 0$ in $\{x \in \mathbb{R}^3 \mid x_3 > 0\} \setminus B_{2R}(\xi_n)$, $0 \leqslant \rho_n \leqslant 1$ and $|\nabla \rho_n| \leqslant 2/R$, and define $\sigma_n \in C^1(\mathbb{R}^3)$ by evenness with respect to the third variable.
Set $v_n = \sigma_n u_n \in H^1_{cyl,o}(\mathbb{R}^3)$ and for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ define

$$\tilde{v}_n(x) = \begin{cases} v_n(x_1, x_2, x_3 + \xi_n - 3R) & \text{if } x_3 > 0 \\ v_n(x_1, x_2, x_3 - \xi_n + 3R) & \text{if } x_3 < 0. \end{cases} \tag{47}$$

We would have that, for $R' = 4R$,

$$\frac{1}{2}\int_{B_{R'}} |\nabla \tilde{v}_n|^2 + \int_{B_{R'}} G_2(\tilde{v}_n) + \frac{q}{4}\int_{B_{R'}} \phi_{\tilde{v}_n}\tilde{v}_n^2 \to c \tag{48}$$

and it is easy to verify also that a sequence so defined is such that

$$\int_{\mathbb{R}^3} G(\tilde{v}_n) \to 1 \quad \text{and} \quad J_q(\tilde{v}_n) \to m_q.$$

So, in any case, by Theorem 2.4 we are able to obtain a minimizing sequence that we label $(u_n)_n$ for the functional restricted to $\mathcal{M}$, which concentrates on a ball centered at the origin and with a sufficiently large radius.
By boundedness of the sequence, we can extract a subsequence weakly convergent in H^1-norm to a function u.
As a consequence of the weak convergence, the Fatou lemma and the weak lower semicontinuity of $\|\nabla \cdot\|_2$,, we have

$$J_q(u) \leqslant \liminf_n J_q(u_n) = m_q. \tag{49}$$

Since we also have

$$u_n \to u \text{ pointwise} \tag{50}$$

$$u_n \to u \text{ in } L^q(B), \text{ for any bounded set } B \text{ and any } q \in [1,6[, \tag{51}$$

we deduce that $u \in H^1_{cyl,o}(\mathbb{R}^3) \setminus \{0\}$ and $G_1(u_n(x)) \to G_1(u(x))$ for any $x \in \mathbb{R}^3$.
Since

$$G_1(s) = o_n(s^2 + |s|^{p+1}) \text{ for } s \to 0 \text{ and } s \to \infty,$$

and by concentration we have

$$\int_{\mathbb{R}^3 \setminus B_R} u_n^2 + |u_n|^{p+1} \to 0,$$

by standard compactness argument (see for instance the proof of Theorem A.I. in the Appendix in [8]) we deduce that

$$\int_{\mathbb{R}^3} G_1(u_n) \to \int_{\mathbb{R}^3} G_1(u).$$

On the other hand, we also have that

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \to \int_{\mathbb{R}^3} G_1(u)$$

and then, by (50)

$$\int_{\mathbb{R}^3} G_2(u) \leqslant \liminf_n \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u) - 1.$$

that is $\int_{\mathbb{R}^3} G(u) \geqslant 1$. We deduce that $\int_{\mathbb{R}^3} G(u) = 1$, otherwise we set $\bar{u} = u(K\,\cdot) \in \mathcal{M}$ with $K = \sqrt[3]{\int_{\mathbb{R}^3} G(u)} > 1$ and by (49) we have,

$$\begin{aligned} m_q \leqslant J_q(\bar{u}) &= \frac{1}{2\sqrt[3]{K}} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4\sqrt[3]{K^5}} \int_{\mathbb{R}^3} \phi_u u^2 \\ &< J_q(u) \leqslant m_q \end{aligned}$$

which is a contradiction.
So $\int_{\mathbb{R}^3} G(u) = 1$, and by (49) $J_q(u) = m_q$. □

Proof of Theorem 1.2 Let $\bar{u} \in \mathcal{M}$ be such that $J_q(\bar{u}) = m_q$ and let $\lambda \in \mathbb{R}$ be the Lagrange multiplier. To show that $\lambda > 0$, we can proceed as in [8, pg 327]. Now define $\tilde{u}$ and $\tilde{\phi}$ as in (18). We prove that $(\tilde{u}, \tilde{\phi})$ satisfies the second equation of the system (19)

$$\begin{aligned} -\Delta\tilde{\phi} &= -\frac{1}{\lambda}\Delta\phi_{\bar{u}}(\cdot/\sqrt{\lambda}) \\ &= \frac{1}{\lambda} q\bar{u}^2(\cdot/\sqrt{\lambda}) \\ &= q'\bar{u}^2(\cdot/\sqrt{\lambda}) = q'\tilde{u}^2. \end{aligned}$$

We prove that $(\tilde{u}, \tilde{\phi})$ satisfies the first equation of the system (19)

$$\begin{aligned} -\Delta\tilde{u} &= -\frac{1}{\lambda}\Delta\bar{u}(\cdot/\sqrt{\lambda}) \\ &= -\frac{1}{\lambda} q\phi_{\bar{u}}(\cdot/\sqrt{\lambda})\bar{u}(\cdot/\sqrt{\lambda}) + g(\bar{u}(\cdot/\sqrt{\lambda})) \\ &= -q'\tilde{\phi}\tilde{u} + g(\tilde{u}) \end{aligned}$$

□

Proof of Theorem 0.1 Let (u, ϕ) be a solution found by Theorem 1.2. The symmetry properties derive from the natural constraint where we have studied the functional of the action and (15).
Now, observe that u can be assumed nonnegative in the semispace $x_3 > 0$ and nonpositive in the semispace $x_3 < 0$.
In fact, if $\bar{u}$ is a minimizer obtained as in Theorem 3.1, we can replace it with the function

$$v = \begin{cases} |\bar{u}| & \text{on } \mathbb{R}^2\times]0, +\infty[; \\ -|\bar{u}| & \text{on } \mathbb{R}^2\times] - \infty, 0[. \end{cases}$$

Obviously $v \in H^1_{cyl,o}(\mathbb{R}^3)$ and since J_q and G are even, v is also a minimizer of $J_q|_{\mathcal{M}}$.
Now we can apply the strong maximum principle in the second equation, and obtain that $\phi > 0$, and in the first equation, obtaining that u can vanish only on the plane $x_3 = 0$. The same considerations on the sign hold for $(\tilde{u}, \tilde{\phi})$, and are true everywhere, since by a standard regularity argument, we can prove that $\tilde{u}$ and $\tilde{\phi}$ are in $C^{2,\alpha}_{loc}(\mathbb{R}^3)$, with $\alpha \in (0,1)$.

□

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